# Computing threshold functions by depth-3 threshold circuits with smaller thresholds of their gates [1]

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**Abstract**

We consider depth-3 unbounded fanin threshold circuits. Gates are usual threshold functions $T_k^n$ which compute 1 iff at least $k$ of the inputs are equal to 1; the minimum $\min\{k, n-k+1\}$ is the *threshold value* of this gate. We show that the function $T_k^n$ cannot be computed by a small depth-3 threshold circuit with threshold values of its gates much smaller than $k$.

*Keywords:* Computational complexity; threshold circuits; lower bounds



## 1 Introduction and results

The $r$th threshold function $T_r^n$ is the boolean function that takes the value 1 precisely when at least $r$ of all $n$ inputs to this function take value 1. Threshold functions play an important role in the investigation of the computational complexity of boolean functions. Their complexity has been studied in various circuit models (cf. [1,8]). Here we are primarily interested in the following problem:

◦ *How efficiently can we compute the function $T_r^n$ by a depth-3 threshold circuits with thresholds of their gates smaller than $r$?*

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We will consider usual (unweighted) threshold circuits. Inputs are variables and their negations together with constants 0 and 1; gates are unbounded fanin threshold gates $T_s^m$; here $s$ is the *threshold*, $m-s+1$ is the *anti–threshold* and their minimum $\min\{s, m-s+1\}$ is the *threshold value* of this gate. We will say that a circuit is *threshold-$s$ circuit* if all its gates have threshold value at most $s$.

The function $T_r^n$ has obvious depth-2 threshold-1 circuit (namely – the OR of ANDs) of size $\binom{n}{r}+1 \leq e^{r(1+\ln(n/r))}$. Khasin [4] has proved that one can do better when using depth-3. He proves (via counting argument) that the function $T_r^n$ can be computed by depth-3 threshold-1 circuit of size $e^r n \ln n$ and bottom fanin $n/r$. The desired circuit is defined as the OR of $e^r$ copies of the random circuit

$$(\vee_{i\in S_1} x_i) \wedge (\vee_{i\in S_2} x_i) \wedge \cdots \wedge (\vee_{i\in S_r} x_i)$$

where $S_1, \ldots, S_k$ is the random partition of $\{1, \ldots, n\}$ into $r$ blocks of size $n/r$. For small values of $r$, namely, when $r \leq \ln n$, this bound was improved by Radhakrishnan [5] to $e^{\sqrt{r}\ln r} n \ln n$.

Can we do much better using threshold-$s$ gates with $s \gg 1$? In this note we show that (up to some restrictions on the bottom gates, i.e gates next to the inputs) the answer is *no*. We will first consider the situation when bottom gates are *arbitrary* Boolean functions but their fanin is restricted.

**Theorem 1** *Let $r, s, t$ and $n$ be positive integers, $s < r$. Any depth-3 threshold-$s$ circuit which computes $T_r^n$ and has arbitrary Boolean functions of fanin at most $t$ at the bottom, must be of size at least $\left(\frac{n}{rdt}\right)^{r/s}$ where $d$ is the maximum anti–threshold of a gate on the middle level.*

Our second result concerns slightly different model. Namely, we allow at the bottom level only ANDs or only ORs but do not place any restriction on their fan-in.

**Theorem 2** *Any depth-3 threshold-$s$ circuit which computes the majority function $T_{n/2}^n$ and has only ORs (respectively, only ANDs) at the bottom, must be of size $\exp\left(\Omega\left(\sqrt{\frac{n}{sd}}\right)\right)$ where $d$ is the maximum anti–threshold (respectively, threshold) of a gate on the middle level.*

We obtain these bounds using the the top-down approach of [3]. It is also possible to derive similar lower bounds from known bounds on the size of $AC^0$-circuits for the majority function proved in [2]. However, such a proof would be based on Switching Lemma which makes the whole argument complicated, whereas the top-down proof uses only elementary combinatorics.

Let us say few words on the power of the model we are dealing with. Even with restricted gates on the bottom, depth-3 threshold circuits have somehow unexpected computational power: Yao in [9] has proved that the whole $\mathrm{ACC}^0$ (which consists of functions computable by polynomial-size constant-depth circuits over the basis $\{\wedge, \vee, \neg, \pmod m\}$ for an arbitrary but fixed $m$) is doable by depth-3 threshold circuits of size $\exp\left((\log n)^{O(1)}\right)$ with AND gates of fanin at most $(\log n)^{O(1)}$ at the bottom. It is therefore hard to prove exponential lower bounds even for this restricted model of threshold circuits: the best lower bound is an $n^{\Omega(\log n)}$ bound proved by Razborov and Wigderson [6] for a depth-3 threshold circuit computing an explicit $\mathrm{ACC}^0$ function, under the restriction that it has either fanin $n^{1-\epsilon}$ gates, or unbounded fanin AND gates at the bottom.

## 2 The proofs

A vector $b \in \{0,1\}^n$ is a *$k$-limit* of a set of vectors $A \subseteq \{0,1\}^n$ if for every subset of $k$ coordinates $S \subseteq [n] \rightleftharpoons \{1, \ldots, n\}$, $|S| = k$, there exists a vector $a \in A$ such that $a \neq b$ but $a$ coincides with $b$ on $S$; it is a *lower* limit if moreover $a \geq b$. Let $\lim_k(A)$ denote the set of all lower $k$-limits of $A$. Let $E_r^n$ denote the $r$-th slice of the cube $\{0,1\}^n$, i.e. the set of all vectors with $r$ ones.

**Lemma 3** ([3]) *Let $A \subseteq E_r^n$. If $|A| > k^r$ then* $\lim_k(A) \neq \emptyset$.

Let $A, B$ be two disjoint subsets of $\{0,1\}^n$. We say that a function $f$ *separates $A$ from $B$* if $f$ outputs 1 on all the vectors in $A$ and outputs 0 on all the vectors in $B$.

**Lemma 4** *Let $C$ be a depth-3 threshold-$s$ circuit of size $\ell$ computing $T_r^n$, and $d$ be the maximum anti–threshold of a gate on the middle level of $C$. Then the OR of at most $d$ bottom gates separates a subset $A \subseteq E_r^n$ from some of its lower $k$-limits where $k = k(n, \ell, r, s)$ is the largest integer not exceeding* $\left[\binom{n}{r} \Big/ \binom{\ell}{s} - 1\right]^{1/r}$.

**Proof.** Since the top gate of $C$ has threshold value at most $s$, the AND $f = \wedge_{g \in G} g$ of some gates on the middle level separates some $\binom{\ell}{s}^{-1}$-fraction $A$ of the $r$-th slice $E_r^n$ from the set $B = \{b : T_r^n(b) = 0\}$. This fraction has more than $k^r$ vectors, so by Lemma 3, the set $B$ contains at least one lower $k$-limit $b$ of $A$. Take a gate $g \in G$ for which $g(b) = 0$. Since $g(A) \equiv 1$, this gate separates $A$ from $b$. The gate $g$ is a threshold gate $T_p^m$ of fanin $m \leq \ell$ and anti–threshold $m - p + 1 \leq d$. That is, $g$ computes 0 if an only if at least $m - p + 1$ of bottom gates feeding in $g$ compute 0. Since $g(b) = 0$, there must be a set $H$ of $m-p+1 \leq d$ bottom gates feeding in $g$ such that $h(b)=0$ for all $h \in H$. Since $g(a)=1$ for each $a \in A$, on each of these vectors at least one $h \in H$ must compute 1. Thus, the OR $\vee_{h \in H} h$ of these gates separates the set $A$ from its lower $k$-limit $b$, as desired. ▯

**Proof of Theorem 1** By Lemma 4, the OR $h$ of at most $d$ bottom gates must separate some set $A$ from some of its its lower $k$-limits $b$ for $k=k(n,\ell,r,s)$. Since bottom gates have fanin at most $t$, the function $h$ depends on at most $dt$ variables which means that $dt > k$ since otherwise $b \in \lim_k(A)$ would imply that $h(a)=h(b)$ for some $a \in A$. We have therefore that $dt \geq |S| > k(n,\ell,r,s)$ which gives the bound $\binom{\ell}{s} \geq \binom{n}{r} \cdot (dt)^{-r}$, and hence, the desired lower bound $\ell \geq \left(\frac{n}{rdt}\right)^{r/s}$ on the size $\ell$. ▯

For the proof of our second theorem we need the following lemma which allows one to kill large fanin gates at the bottom.

**Lemma 5** **([3])** *Let $\mathcal{F}$ be a family of less than $\left(\frac{n+1}{m+1}\right)^t$ subsets of $[n]$ each of cardinality more than $t$. Then there exists a subset $T \subseteq [n]$ such that $|T| \leq n-m$ and $T$ intersects every set in $\mathcal{F}$.*

**Proof of Theorem 2** Let $C$ be a depth-3 threshold-$s$ circuit which computes the majority $T^n_{n/2}$ and has only ORs or only ANDs at the bottom. Since the majority function is self-dual, the dual of $C$ also computes the majority. So, it is enough to prove the theorem in case of ORs at the bottom.

Let $d$ be the maximum anti–threshold of a gate on the middle level of $C$. Let $\ell$ be the size of $C$ and suppose that $\ell < \left(\frac{n+1}{m+1}\right)^t$ where $m$ and $t$ are parameters to be specified later. Take $\mathcal{F}$ to be the family of the sets of (indices of) unnegated inputs to the OR gates at the bottom of $C$. Then $|\mathcal{F}| \leq \ell$ and by Lemma 5, we can replace at most $n-m$ variables by the constant 1 so that each of remaining OR gates at the bottom of the resulting circuit $C'$ has at most $t$ positive literals, i.e all but $t$ variables in each remaining OR must be negated. This new circuit $C'$ computes the threshold function $T^m_r$ with $r=m-n/2$.

By Lemma 4, the OR $h$ of at most $d$ bottom gates separates some set $A$ from some of its lower limits $b \in \lim_k(A)$ for $k=k(m,\ell,r,s)$. The function $h$ is simply an OR of literals. Let $S$ be the set of positive (i.e unnegated) literals in it. Then $|S| \leq dt$. We claim that $|S| > k$. To verify this, assume that $|S| \leq k$, and let $h^+$ ($h^-$) be the OR of all positive (resp. negative) literals in $h$. Since $b$ is a $k$-limit of $A$ and $|S| \leq k$, there is a vector $a \in A$ which coincides with $b$ on all the variables in $S$, and hence, $h^+(a)=h^+(b)=0$. Moreover,

$a \geq b$ since $b$ is a lower limit, and hence, $h^-(a) \leq h^-(b) = 0$. Thus, $h(a) = 0$, which means that $h$ does not separate $a$ from $b$, a contradiction. Therefore, $dt \geq |S| > k(m, \ell, r, s)$ which gives the bound $\ell \geq \left(\frac{m}{rdt}\right)^{r/s}$.

It remains to choose appropriate values for the parameters $m$ and $t$. Since $r = m - \frac{n}{2}$, we would like to take $m$ to be the largest integer such that $m - \frac{n}{2} \leq \frac{m}{edt}$, which would give the bound $\ell \geq e^{r/s}$ with $r = \left\lfloor \frac{m}{edt} \right\rfloor$. So take $m \rightleftharpoons \left\lfloor \frac{n}{2\alpha} \right\rfloor$ where $\alpha = \alpha(t) = 1 - \frac{1}{edt}$. Given $m$, the parameter $t$ must fulfill $\ell < \left(\frac{n+1}{m+1}\right)^t$, so we must be sure that $e^{r/s} < \left(\frac{n+1}{m+1}\right)^t$. By the choice of $m$, $\frac{n+1}{m+1} \geq 2\alpha(t) > 4/3$ and, since $r \leq \frac{m}{edt}$, it is enough to ensure that $\frac{m}{esdt} \leq t \ln(4/3)$, which holds for $t \rightleftharpoons \left\lfloor \sqrt{\frac{m}{esd \ln(4/3)}} \right\rfloor$. Therefore, the lower bound $\ell \geq e^{r/s}$ holds with $r = \left\lfloor \frac{m}{edt} \right\rfloor = \Omega\left(\sqrt{\frac{sn}{d}}\right)$, which gives the desired lower bound $\ell \geq \exp\left(\Omega\left(\sqrt{\frac{n}{sd}}\right)\right)$. ∎


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