$$\Big|\sum_{x\in\mathbb{F}_q}\chi\big(f_C(x)\big)\Big| \le (|C|-1)\sqrt{q}.$$

Hence,

$$\Big|\sum_{x\in\mathbb{F}_q} g(x) - q\Big| \le \sum_C (|C|-1)\sqrt{q} = \sqrt{q}\sum_{s=2}^{k}\binom{k}{s}(s-1)$$
$$= \sqrt{q}\big((k-2)2^{k-1}+1\big).$$

Here the last equality follows from the identity $\sum_{s=1}^{k} s\binom{k}{s} = k2^{k-1}$ (see Exercise 1.5).

The summation above is over all nodes $x \in V_2 = \mathbb{F}_q$. However, for every node $x \in A' \cup B'$, $g(x) \le 2^{k-1}$, and the nodes of $A' \cup B'$ can contribute at most

$$\Big|\sum_{x\in A'\cup B'} g(x)\Big| \le k\cdot 2^{k-1}.$$

Therefore,

$$\Big|\sum_{x\in U} g(x) - q\Big| \le \sqrt{q}\big((k-2)2^{k-1}+1\big) + k\cdot 2^{k-1}.$$

Dividing both sides by 2^k and using (10.7), together with the obvious estimate $v(A,B) - v^*(A,B) \le |A'\cup B'| = k$, we conclude that

$$\Big|v(A,B) - 2^{-k}q\Big| \le \frac{k\sqrt{q}}{2} - \sqrt{q} + \frac{\sqrt{q}}{2^k} + \frac{k}{2} + k, \tag{10.8}$$

which does not exceed $k\sqrt{q}$ as long as $q \ge 9$. □

Theorem 10.20 together with Proposition 10.19 give us, for infinitely many values of n, and for every k such that $k2^k < \sqrt{n}$, an explicit construction of (n,k)-universal sets of size n. In Sect. 17.4 we will show how to construct such sets of size $n^{O(k)}$ for arbitrary k using some elementary properties of linear codes.

10.7 Full graphs

We have seen that universal sets of 0-1 strings correspond to bipartite graphs satisfying the isolated neighbor condition. Let us now ask a slightly different question: how many vertices must a graph have in order to contain *every* k-vertex graph as an induced subgraph? Such graphs are called *k-full*. That is, given k, we are looking for graphs of small order (the order of a graph is the number of its vertices) which contain every graph of order k as an induced subgraph.

Note that if G is a k-full graph of order n then $\binom{n}{k}$ is at least the number of non-isomorphic graphs of order k, so

$$\binom{n}{k} \geq 2^{\binom{k}{2}}/k!$$

and thus

$$n \geq 2^{(k-1)/2}.$$

On the other hand, for every k it is possible to exhibit a k-full graph of order $n = 2^k$. This nice construction is due to Bollobás and Thomason (1981).

Let P_k be a graph of order $n = 2^k$ whose vertices are subsets of $\{1, \ldots, k\}$, and where two distinct vertices A and B are joined if and only if $|A \cap B|$ is even; if one of the vertices, say A, is an empty set then we join B to A if and only if $|B|$ is even. Note that the resulting graph is regular: each vertex has degree $2^{k-1} - 1$.

Theorem 10.22 (Bollobás–Thomason 1981)**.** *The graph P_k is k-full.*

Proof. Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_k\}$. We claim that there are sets $A_1, A_2, \ldots, A_k$ uniquely determined by G, such that

$$A_i \subseteq \{1, \ldots, i\}, \quad i \in A_i,$$

and, for $i \neq j$,

$$|A_i \cap A_j| \text{ is even if and only if } v_i \text{ and } v_j \text{ are joined in } G.$$

Indeed, suppose we have already chosen the sets $A_1, A_2, \ldots, A_{j-1}$. Our goal is to choose the next set A_j which is properly joined to all the sets $A_1, A_2, \ldots, A_{j-1}$, that is, $|A_j \cap A_i|$ must be even precisely when v_j is joined to v_i in G. We will obtain A_j as the last set in a sequence $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_{j-1} = A_j$, where, for each $1 \leq i < j$, B_i is a set properly joined to all sets $A_1, A_2, \ldots, A_i$.

As the first set B_1 we take either $\{j\}$ or $\{1, j\}$ depending on whether v_j is joined to v_1 or not. Having the sets $B_1, \ldots, B_{i-1}$ we want to choose a set B_i. If v_j is joined to v_i then we set $B_i = B_{i-1}$ or $B_i = B_{i-1} \cup \{i\}$ depending on whether $|B_{i-1} \cap A_i|$ is even or odd. If v_j is not joined to v_i then we act dually. Observe that our choice of whether i is in B_i will effect $|B_i \cap A_i|$ (since $i \in A_i$) but none of $|B_i \cap A_l|$, $l < i$ (since $A_l \subseteq \{1, \ldots, l\}$). After $j - 1$ steps we will obtain the desired set $B_{j-1} = A_j$. $\square$