Assignment 11

Note: It is understood that all of your statements have to be proven correct.

Note: Solutions may be submitted by email. Solutions submitted after the lecture will not be graded.

Note: The maximum score for this assignment is 16. All exceeding points will be added to the total score of all assignments.

Exercise 11.1. (8)

A Top-10 algorithm

We want to sample from a finite state space $S$. We construct a Markov Chain $C$ on $S$ such that the stationary distribution of a random walk on $S$ coincides with a given distribution $\pi$, where $\pi_s > 0$ for all $s \in S$. (For instance we may choose $S$ to be the set of independent sets of a given graph and $\pi$ the uniform distribution on these independent sets).

To construct $C$ we define a neighborhood $N(s) \subseteq S$ for all $s \in S$ such that the resulting graph is connected. To complete the construction for $C$ define the transition matrix by

$$P(s, t) = \begin{cases} \frac{1}{d} \cdot \frac{1}{2} \cdot \min(1, \frac{\pi(t)}{\pi(s)}) & \text{if } t \in N(s), \\ \text{the remaining probability} & \text{if } t = s. \end{cases}$$

where $d = \max_{s \in S} |N(s)|$.

Prove that $\pi$ is indeed $C$’s unique stationary distribution.

Exercise 11.2. (16)

Mixing and Coupling

We investigate how fast a random walk on an ergodic Markov Chain $C$ with state space $S$ approximates its stationary distribution. Therefore we measure the distance of a $k$-step walk to $C$’s stationary distribution $\pi$ by

$$||P^k, \pi|| = \max_{s \in S} \frac{1}{2} \sum_{t \in S} |P^k(s, t) - \pi(t)|.$$  

Utilizing this distance we now define the $\epsilon$-mixing time of $C$ to be

$$m(\epsilon) = \min\{k : \forall k' \geq k : ||P^{k'}, \pi|| \leq \epsilon\}.$$  

A Markov Chain is said to be rapidly mixing if $m(\epsilon) = O(\text{polylog}(|S|) \cdot \ln(1/\epsilon))$.

We now discuss the method of coupling to bound mixing times. A coupling $\zeta$ for the Markov chain $C$ is a random walk on the state space $S \times S$. In particular, for all starting states $(s, t)$ of $\zeta$, the transition probabilities must satisfy the following conditions:
• If \((X_k, Y_k)_{k=0}^{\infty}\) is the sequence of random variables for states of the coupling \(C\), then \((X_k)_{k=0}^{\infty}\) and \((Y_k)_{k=0}^{\infty}\) are sequences of random variables for the states of walks defined by the original chain \(C\) starting in \(s\) and \(t\), respectively.

• Whenever \(X_k\) and \(Y_k\) coincide, so do \(X_{k+1}\) and \(Y_{k+1}\).

Define the coupling time \(T_{s,t} = \min\{k \mid X_k = Y_k\text{ if } C\text{ starts in } (s, t)\}\). You may assume that

\[
m(\epsilon) \leq e \cdot \ln(1/\epsilon) \cdot \max_{s,t \in S} E(T_{s,t}),
\]

holds, where \(e\) is Euler’s number.

We want to bound the mixing time for the following Markov chain defining random walks on the \(n\)-dimensional hypercube \(H_n\). In the current state \(x_1 \ldots x_n\) choose a random position \(i \in \{1, \ldots, n\}\) and a random bit \(b\) and jump to state \(x_1 \ldots \hat{x}_i \ldots x_n\) if \(b = 1\) and \(x_1 \ldots \hat{x}_i \ldots x_{n-1} \oplus b x_n \ldots x_{n-1}\).

Define a coupling for the chain such that \(\max_{s,t \in S} E(T_{s,t})\) is small. In particular show that the chain is rapidly mixing.

\textit{Hint:} Observe that whenever the two walks reach the same state at the same time \(k\), then \(X_{k+1} = Y_{k+1}\).